Winter School

Analytical Methods

in

Quantum and Continuum Mechanics

Turin, November 29th – December 3rd 2021

ANALYTICAL METHODS IN QUANTUM AND CONTINUUM MECHANICS

	Monday	Tuesday	Wednesday	Thursday	Friday
9:00 - 10:00	Welcome and Registration	**Marco CICALESE** Topological singularities in classical spin systems: a variational perspective	**Marco CICALESE** Topological singularities in classical spin systems: a variational perspective	**Andrea MALCHIODI** Variational theory of Liouville equations	**Marco CICALESE** Topological singularities in classical spin systems: a variational perspective
10:00 - 11:00	**Nicola VISCIGLIA** The Nonlinear Schrödinger Equation with spatial white noise				
11:00 - 11:20	Coffee Break	Coffee Break	Coffee Break	Coffee Break	Coffee Break
11:20 - 12:20	**Nicola VISCIGLIA** The Nonlinear Schrödinger Equation with spatial white noise	**Nicola VISCIGLIA** The Nonlinear Schrödinger Equation with spatial white noise	**Andrea MALCHIODI** Variational theory of Liouville equations	**Nicola VISCIGLIA** The Nonlinear Schrödinger Equation with spatial white noise	**Andrea MALCHIODI** Variational theory of Liouville equations
12:20 - 13:20	**Giuseppe SAVARE'** Evolution of probability measures and optimal transport				
13:20 - 14:50	Lunch	Lunch	Lunch	Lunch	Lunch
14:50 - 15:50	**A. Audrito** (14:50 - 15:15) **R. Scandone** (15:15 - 15:40)	**Flaviana IURLANO** Shape optimization of light structures and the vanishing mass conjecture	Free afternoon	**Aleks JEVNIKAR** Existence results for super-Liouville equations	
15:50 - 16:05	Coffee Break	Coffee Break		Coffee Break	
16:05 - 16:30	**A. Bach**	**F. Boni**		**G. Bevilacqua**	
16:30 - 16:55	**F. De Vecchi**	**L. Hientzsch**		**L. D'Elia**	
16:55 - 17:20	**P. Rinaldi**	**M. Jex**		**F. Dipasquale**	
17:20 - 17:45	**I. Zachhuber**	**M. Schulz**		**P. Wozniak**	

social dinner on Wednesday night

Courses

Topological singularities in classical spin systems: a variational perspective

Marco Cicalese
Technische Universität München

Typical low–energy states of lattice spin systems show the emergence of complex structures at large scales. The analysis of such structures, their fine geometry and energetic behaviour is the main goal of this course. Due to the multiscale nature of spin systems, their rigorous mathematical study poses a number of very challenging problems. Their analysis often requires the combination of different analytical, stochastic, geometric, and combinatorial techniques. This course aims at introducing a possible variational approach to the analysis of some energy driven lattice systems, mainly focusing on those energy scalings which lead to the emergence of topological singularities. After a quick review of the necessary preliminary results from Geometric Measure Theory and the Direct Methods in the Calculus of Variations, I will discuss the large–scale behaviour of the ferromagnetic and anti–ferromagnetic xy systems on (possibly geometrically frustrated) lattices.

Variational theory of Liouville equations

Andrea Malchiodi
Scuola Normale Superiore Pisa

Liouville equations play a fundamental role in Geometry when prescribing the Gaussian curvature of a (possibly singular) surface, as well as in models from Mathematical Physics that describe stationary Euler flows, Electro-weak, Chern-Simons models of superconductivity and String Theory. We will attack the existence issue by exploiting the variational structure of the problem. In general, global minimizers may not exist, and we will focus on the existence of saddle-type solutions constructed via min-max theory. A crucial tool in doing this will be to find suitable improvements of the Moser–Trudinger inequality by a fine analysis of the distribution of volume over the surface. Some applications to Functional Determinants in spectral theory will be also described.

The nonlinear Schrödinger equation with spatial white noise

Nicola Visciglia
Università di Pisa

We discuss the existence and uniqueness of global solutions almost surely for NLS posed on T^2 and perturbed by a spatial white noise. The main idea is to use a gauge transform, first used by M. Hairer in the parabolic setting, in order to work in a more regular framework and then to renormalize the corresponding stochastic objects. Once this probabilistic procedure is concluded we shall show how to get deterministic uniform a priori bounds on the corresponding solutions and how to pass to the limit. In particular we shall consider the cubic and fourth order nonlinearity. The cubic case was first achieved by A. Debbusche and H. Weber, the quartic case is treated in a joint work of the speaker with N. Tzvetkov.

Main Talks

Shape optimization of light structures and the vanishing mass conjecture

Flaviana Iurlano
Sorbonne Université

We present rigorous results about the vanishing-mass limit of the classical problem to find a shape with minimal elastic compliance. Contrary to all previous results in the mathematical literature, which utilize a soft mass constraint by introducing a Lagrange multiplier, we here consider the hard mass constraint. Our results are the first to establish the convergence of approximately optimal shapes of (exact) size $\varepsilon \to 0$ to a limit generalized shape represented by a (possibly diffuse) probability measure. This limit generalized shape is a minimizer of the limit compliance, which involves a new integrand, namely the one conjectured by Bouchitté in 2001 and predicted heuristically before in works of Allaire & Kohn and Kohn & Strang from the 1980s and 1990s. This integrand gives the energy of the limit generalized shape understood as a fine oscillation of (optimal) lower-dimensional structures. Its appearance is surprising since the integrand in the original compliance is just a quadratic form and the non-convexity of the problem is not immediately obvious. In fact, it is the interaction of the mass constraint with the requirement of attaining the loading (in the form of a divergence-constraint) that gives rise to this new integrand. Our proofs rest on compensated compactness arguments applied to an explicit family of (symmetric) div-quasiconvex quadratic forms, computations involving the Hashin-Shtrikman bounds for the Kohn-Strang integrand, and the characterization of limit minimizers due to Bouchitté & Buttazzo. This is a joint work with J.-F. Babadjian and F. Rindler.

Existence results for super-Liouville equations

Aleks Jevnikar
Università degli Studi di Udine

We consider super-Liouville equations on closed surfaces, which have a variational structure with a strongly-indefinite functional. We obtain the first existence results by making use of min-max methods and bifurcation theory. Joint project with Andrea Malchiodi and Ruijun Wu.

Evolution of probability measures and optimal transport

Giuseppe Savaré
Università Bocconi – Milano

Starting from the celebrated papers by Jordan, Kinderlehrer and Otto and by Benamou-Brenier, the connection between many important evolutionary PDE's and the dynamic formulation of Optimal Transport has been deeply investigated. By using powerful methods of metric and functional analysis, measure theory, and calculus of variations, a general theory of non-smooth differential calculus and gradient flows in the Wasserstein spaces have been developed, together with some general tools (as the representation results for the solution of the continuity equation) which have been applied to many different contexts. We will give a brief review of these developments and some of the main results in this field.

Contributed Talks

A variational approach to a class of nonlinear Cauchy-Neumann problems

Alessandro Audrito
ETH Zürich

We prove existence and Hölder regularity of weak solutions to a class of nonlinear Cauchy-Neumann problems arising in combustion theory and fractional diffusion. Weak solutions are obtained through a nonstandard variational approximation procedure, known in the literature as the WIDE method. Our main contribution is to establish a uniform Hölder estimate of De Giorgi-Nash-Moser type for the approximating sequence.

Stochastic homogenisation of singularly-perturbed elliptic functionals

Annika Bach
Università degli Studi di Roma La Sapienza

Starting from the seminal works by Modica and Mortola and by Ambrosio and Tortorelli, singularly-perturbed elliptic functionals have proven to be an efficient tool to approximate free-discontinuity problems in various situations. At the same time, these kind of functionals provide instances of gradient-damage models. In this talk we are interested in the effective behaviour of singularly-perturbed elliptic functionals in the presence of random heterogeneities. More in detail, we consider random Ambrosio-Tortorelli type functionals of the form

$$F_\varepsilon(\omega)(u,v) = \int_D v^2 f\left(\omega, \frac{x}{\varepsilon}, \nabla u\right) dx + \int \frac{(1-v)^2}{\varepsilon} + g\left(\omega, \frac{x}{\varepsilon}, \nabla_v\right) dx, \qquad (u,v) \in W^{1,2}(D),$$

and we aim at characterising their Γ-limit as$\varepsilon \to 0$. Assuming f and g to be stationary, of superlinear growth in the gradient variable, and satisfying mild regularity assumptions, we show that the Γ-limit is a homogenised random free-discontinuity functional of brittle type. To achieve such a result, we combine the abstract localisation method of Γ-convergence with subadditive ergodic theorems. In this procedure, a crucial step consists in proving that bulk and surface contributions decouple in the limit process. This is joint work with R. Marziani and C. I. Zeppieri (Münster).

The Kirchhoff-Plateau problem and its generalizations

Giulia Bevilacqua
Università Cattolica del Sacro Cuore

Joint work with Luca Lussardi (DISMA – Politecnico di Torino, luca.lussardi@polito.it) and Alfredo Marzocchi (Universit`a Cattolica del Sacro Cuore, alfredo.marzocchi@unicatt.it).

The Kirchhoff-Plateau problem concerns the equilibrium shapes of a system in which a flexible filament in the form of a closed loop is spanned by a liquid film, with the filament being modeled as a Kirchhoff rod and the action of the spanning surface being solely due to surface tension. Giusteri, Lussardi and Fried in [6] established the existence of an equilibrium shape that minimizes the total energy of the system under the physical constraint of noninterpenetration of matter, but allowing for points on the surface of the bounding loop to come into contact. In [1, 2], we use this result to generalize the situation studying a system composed by several rods linked in an arbitrary way and tied by a soap film and we perform some experiments to validate our result. We also study the Elastic Plateau problem, i.e. the above problem when the boundary is an elastic curve. In [3], we obtain the minimal energy solution of the Plateau problem with elastic boundary as a variational limit of the minima of the Kirchhoff-Plateau problems with a rod boundary when the cross-section of the rod vanishes. The limit boundary is a framed curve that can sustain bending and twisting.

Finally, since computing the minimum of the energy functional is quite easy, we aim at characterizing the critical points, i.e. computing all the equilibrium configurations to complete understand the mechanical structure of the Plateau problem. First, in [4], we propose a variation of the Lagrange multiplier theorem in infinite dimension, obtaining a non-homogeneous first order differential system of equations for the only elastic curve. Then, in [5], we derive the Euler-Lagrange equations for the coupled system elastic rod and soap film surface.

References.

[1] G. Bevilacqua, L. Lussardi, A. Marzocchi, Soap film spanning electrically repulsive elastic protein links, Proceedings of School & Research Workshop Mathematical Modeling of Self-Organizations in Medicine, Biology and Ecology: from micro to macro, Atti Accad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur. 96 (2018), suppl. 3, A1, 13pp.
[2] G. Bevilacqua, L. Lussardi, A. Marzocchi, Soap film spanning an elastic link, Quart. Appl. Math. 77 (3) (2019), 507–523.
[3] G. Bevilacqua, L. Lussardi, A. Marzocchi, Dimensional reduction of the Kirchhoff-Plateau problem, J. Elasticity 140, 135–148 (2020).

[4] G. Bevilacqua, L. Lussardi, A. Marzocchi, Variational analysis of inextensible elastic curves, arXiv preprint arXiv:2106.01659 (2021).
[5] G. Bevilacqua, L. Lussardi, A. Marzocchi, Euler-Lagrange Equations for the KirchhoffPlateau problem, in preparation.
[6] G.G. Giusteri, L. Lussardi, E. Fried, Solution of the Kirchhoff-Plateau problem, J. Nonlinear Sci. 27 (2017), 1043–1063.

Ground states of the 2D nonlinear Schrödinger equation with a delta interaction

Filippo Boni
Politecnico di Torino

We investigate the existence of ground states at fixed mass of the subcritical nonlinear Schrödinger equation with a delta interaction in dimension two. We prove that ground states exist for every positive mass and show a logarithmic singularity at the interaction point. Moreover, up to a multiplication by a phase factor, they are positive, radially symmetric, and decreasing along the radial direction. In order to overcome the obstacles arising from the uncommon structure of the energy space, we prove an ad hoc result on rearrangements and move to the study of the minimizers of the action functional on the Nehari manifold, relying on its connection with the original problem. This is a joint work with R. Adami, R. Carlone and L. Tentarelli.

Homogenization of discrete thin structure

Lorenza D'Elia
Università degli Studi di Roma Tor Vergata

We combine a discrete-to-continuum and dimension-reduction process to investigate discrete thin objects. We consider quadratic energies defined on a portion X of $\mathbb{Z}^d \times \{0, \dots, M-1\}^k$, for some $M \in \mathbb{N}$. We only require that X is connected as a graph and periodic in the first d-directions. Upon scaling of the domain and the energies by a small parameter ε, the scaled energies Γ-converge to a d-dimensional functional. This is a joint work with A. Braides (Università di Roma Tor Vergata).

Stochastic quantization of exponential-type quantum field theories

Francesco De Vecchi
Universität Bonn

Stochastic quantization is a method, proposed by Parisi and Wu, of constructive Euclidean quantum field theory for building the Schwinger functions of a quantum model from the invariant solutions of suitable (parabolic, hyperbolic or elliptic) stochastic partial differential equations (SPDEs). In the talk we provide an introduction of the topic and of some recent developments in the field, focusing on the analytic and probabilistic aspects of the problem. We propose a more detailed analysis of the SPDEs related to the two-

dimensional exponential-type models such as the Høegh-Krohn, or Liouville quantum gravity, quantum field theory and the massive $\sinh(\varphi)_2$ interaction. The talk is mainly based on the joint work [1] with Sergio Albeverio and Massimiliano Gubinelli, and on [2] with Nikolay Barashkov.

References.

[1] Albeverio, Sergio, Francesco C. De Vecchi, and Massimiliano Gubinelli. "The elliptic stochastic quantization of some two dimensional Euclidean QFTs." arXiv preprint arXiv:1906.11187, to appear in Annales de l'Institut Henri Poincaré.
[2] Barashkov, Nikolay, and Francesco C. De Vecchi. "Elliptic stochastic quantization of Sinh-Gordon QFT." arXiv preprint arXiv:2108.12664 (2021).

Topology of minimizers in the Landau-De Gennes model of liquid crystals

Federico Luigi Dipasquale
Università degli Studi di Verona

I will talk about recent results on the topology of minimizers in the Landau-de Gennes theory of (nematic) liquid crystals. These results, obtained in collaboration with V. Millot and A. Pisante, shed light on a long-standing problem in this theory; i.e., that of "biaxial torus solutions" and of "split solutions". These solutions of the corresponding Euler-Lagrange equations are known since long time through numerical simulations, according to which they are competing minimizers of the energy functional in very relevant physical conditions, but they have so far eluded precise mathematical description. In our works, we introduced some new ideas that led to a description of minimizers depending on the topology of the domain and of the boundary condition that is in remarkable qualitative agreement with numerical simulations. To this purpose, we employed techniques from many areas of mathematics, among which calculus of variations, PDEs, differential and algebraic geometry.

Global well-posedness in the energy space of nonlinear Schrödinger equations with non-vanishing boundary conditions at infinity

Lars Eric Hientzsch
Universität Bielefeld

Nonlinear Schrödinger equations with non-vanishing boundary conditions at infinity arise among others as models in superfluidity, Bose-Einstein condensation and nonlinear optics. Unlike the most usually studied case with trivial far-field, these equations admit a large variety of special solutions, such as travelling waves, even for "defocusing" nonlinearities. We show local well-posedness in the energy space under quite general assumptions on the nonlinearity, analogue to the Kato theory available for the equations with trivial far-field. Our result includes in particular power-type nonlinearities that are in general not covered by previous results. Global well-posedness is proven for non-negative nonlinear potentials. Our approach relies on a detailed analysis of the metric structure of the energy space. Continuous dependence on the initial data w.r.t. the topology induced by the metric on the energy space turns out to be a delicate issue due to the non-trivial far-field behavior. This talk is based on joint work with P. Antonelli and P. Marcati (Gran Sasso Science Institute).

Quantum Systems at The Brink: Critical Potentials and dimensionality

Michal Jex
Université Paris Dauphine

Joint with Dirk Hundertmark, Markus Lange.
The existence of eigenfunctions for Schrödinger operators are of utmost importance in quantum mechanics and its applications. It is well known that for eigenvalues below the threshold of the essential spectrum, eigenvectors exist and decay exponentially. However, the situation at the threshold is much more subtle. We present necessary and sufficient condition for the Schrödinger operator to have zero energy ground state. We show that it critically depends on the asymptotic behaviour of the potential. We derive necessary and sufficient conditions for the existence and absence of zero eigenvalue with respect to the dimension d. We show that the leading order term has a strong dependence on the dimension, namely $\frac{d(4-d)}{|x^2|}$ for $|x| \to \infty$. Furthermore our results are in the mathematical sense sharp.

An Algebraic and Microlocal Approach to Stochastic Non-Linear Schrödinger Equation

Paolo Rinaldi
Universität Bonn

In this talk I will present a generalization of a recent perturbative approach to the study of non-linear stochastic partial differential equations which borrows techniques from algebraic quantum field theory [Commun. Contemp. Math. 2150075]. In particular this framework is applied to the stochastic non-linear Schrödinger equation, allowing for an explicit calculation of expectation values and correlation functions of polynomial expressions of the solution. This procedure leads to a priori ill-defined contributions, a typical phenomenon in the theory of non-linear SPDEs, as well as in quantum field theory. Renormalization of these problematic structures is tackled from a microlocal viewpoint. Finally, the question regarding renormalizability of the model is addressed via techniques proper of graph theory as well as of scaling degree theory.

Existence and stability for a QMHD system

Raffaele Scandone
Gran Sasso Science Institute

In this talk we discuss the solution theory for a quantum magneto-hydrodynamic (QMHD) system, which models the dynamics of a charged quantum fluid, interacting with its self-generated electromagnetic field. By means of the Madelung transform, the QMHD system if formally connected to a Maxwell-Schrödinger system with a pure-power non-linearity. Using this correspondence, and exploiting suitable dispersive estimate for the magnetic NLS, we show the existence of global, finite energy, weak solutions to the QMHD system, as soon as the initial magnetic potential is slightly more regular than finite energy. In the same regime of regularity, we also show a local-smoothing effect for the magnetic NLS, which allows to prove a stability result for the QMHD system. Finally, we briefly discuss the technical obstructions which prevents a theory in the energy space.

On the Maximal Excess Charge of Atoms

Marvin R. Schulz
Karlsruhe Institute for Technology

The question on the maximal ionization of atoms is a long standing open problem in Mathematical Physics and is listed in Simon's Fifteen Problems in Mathematical Physics as Problem 10D. It is conjectured that for atoms (in the vacuum) the number of electrons N can only exceed the number of protons Z by at most 2 electrons. There are mainly two results that give an explicit upper bound on the maximal ionization for any number of protons by Lieb (Lieb, 1984) who showed $N_C < 2Z + 1$ and Nam (Nam, 2017) who showed $N_c < 1.22Z + 3.0Z^{1/3}$. Our results are the third of that kind and generalize and improve the approaches given by Lieb and Nam. The results are relevant for small and large atoms. We improve for $Z \geq 7$ both existing results. In the large Z-Limit we give an alternative proof for the asymptotic neutrality that additionally generalizes to the case of non-vanishing magnetic fields.

High-contrast Mumford-Shah functionals on perforated domains

Piotr Wozniak
University of Münster

We shall discuss the asymptotic behaviour of free-discontinuity functionals that admit volume degeneracies distributed periodically in a prescribed domain at ε-scales. We will explore through Γ-convergence that the homogenisation procedure leads to a so called volume-surface interaction in the limit. Intriguingly considering various perturbation scaling in the bulk energy plays a key role in the resulting cell formulae for the discontinuity part of the Γ-limiting functional. In terms of fracture mechanics, we intend to study the way in which soft periodic inclusions in a material of varying elastic moduli affect formations of cracks.

Finite speed of propagation for multiplicative stochastic wave equation

Immanuel Zachhuber
Freie Universität Berlin

I will prove finite speed of propagation for a renormalised wave equation with spatial white noise potential in 2- and 3- dimensions. This leads to global space-time of the associated cubic multiplicative stochastic wave equation. Due to the irregularity of the noise, some tools from singular SPDEs are required as well as Tartar's approach to finite speed of propagation.